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Microscopic Modes in a Fermi Superfluid. II. Dispersion Relations

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This is the second of two papers in which microscopic expressions for the amplitudes and dispersion relations for hydrodynamic modes in an isotropic Fermi superfluid are derived. In this paper we obtain approximate solutions to the linearized kinetic equations for the bogolon spin density and total density for the case of long-wavelength disturbances after long times when a fluctuating superfluid velocity is present. In so doing, we obtain microscopic expressions for the amplitude and dispersion relations for the spin diffusion mode, the two shear modes, and the four longitudinal modes (two first-sound modes and two second-sound modes).

KEY WORDS: Fermi superfluids; transport theory; hydrodynamic modes; microscopic mode theory; broken symmetry; Wigner functions.

1. INTRODUCTION

In a previous paper, which we shall call I, we derived decoupled linearized kinetic equations for the bogolon spin density and total bogolon density in an inhomogeneous Fermi superfluid with a fluctuating superfluid velocity. In this paper, we shall obtain approximate solutions to the linearized kinetic equations for the case of long-wavelength inhomogeneities and long times. That is, we shall obtain microscopic expressions for the amplitudes and dispersion relations of the spin diffusion mode, the two shear modes, and the four longitudinal modes (two first-sound modes and two second-sound modes).

We begin in Section 2 with the equations for the bogolon spin density and, using perturbation techniques, we derive microscopic expressions for the amplitude and dispersion relations for the spin diffusion mode. At the same time, we obtain a microscopic expression for the coefficient of spin diffusion in terms of a bogolon velocity autocorrelation function.

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In Section 3 we begin with the more difficult problem of obtaining the modes for quantities which depend on the total bogolon density. The shear modes are relatively simple to obtain because the kinetic equation for that case can be written in the form of a linear eigenvalue equation. In Section 3 we obtain microscopic expressions for the shear mode amplitudes and dispersion relations, and we obtain an expression for the coefficient of shear viscosity in terms of a bogolon momentum current autocorrelation function.

In I we were able to close the kinetic equations by requiring that the total particle density obey a continuity equation. This is the usual assumption in the hydrodynamic theory for isotropic superfluids. We found, however, that this method of closure introduced an imaginary term into the expression for the chemical potential. In Section 4 we discuss the effect of this term on the longitudinal modes.

For the case of the longitudinal modes, the kinetic equation is no longer linear in the frequency and wave vector of the inhomogeneities, and therefore it does not take the form of a linear eigenvalue equation. However, we can still use perturbation techniques to find approximate solutions to the longitudinal kinetic equation. As we shall see, it is precisely this nonlinear character which enables us to obtain microscopic expressions for the four longitudinal hydrodynamic modes. In Section 5 we obtain microscopic expressions for the longitudinal mode amplitudes and dispersion relations, and we obtain microscopic expressions for the various sound velocities introduced in I. Finally, in Section 6 we make some concluding remarks.

2. SPIN DIFFUSION

The linearized kinetic equation for the bogolon spin density can be written in the form

$$i\frac{\partial m(\mathbf{k}, \mathbf{q}, t)}{\partial t} - \frac{\hbar\mathbf{k}\cdot\mathbf{q}}{m}\frac{\xi_{\mathbf{k}}}{E_{\mathbf{k}}{}^{0}}(1 - \hat{G}_{\mathbf{k}})m(\mathbf{k}, \mathbf{q}, t) = i\hat{I}_{\mathbf{k}}^{(-)}(1 - \hat{G}_{\mathbf{k}})m(\mathbf{k}, \mathbf{q}, t) \quad (2.1)$$

where the integral operator $\hat{G}_{\mathbf{k}}$ is defined by

$$\hat{G}_{\mathbf{k}}m(\mathbf{k}, \mathbf{q}, t) \equiv \frac{g}{V}\sum_{\mathbf{k}'}\frac{\partial f_{\mathbf{k}'}^{0}}{\partial E_{\mathbf{k}'}^{0}}m(\mathbf{k}', \mathbf{q}, t) \quad (2.2)$$

and the collision operator $\hat{I}_{\mathbf{k}}^{(-)}$ is defined by

$$\hat{I}_{\mathbf{k}}^{(-)} = \frac{1}{\hbar}\left(\frac{\partial f_{\mathbf{k}}{}^{0}}{\partial E_{\mathbf{k}}{}^{0}}\right)^{-1}\hat{\mathbb{C}}_{\mathbf{k}}^{(-)} \quad (2.3)$$

The collision operator $\hat{\mathbb{C}}_{\mathbf{k}}^{(-)}$ was defined in Eq. (I.4.29). In order to simplify our notation, it is convenient to introduce the following scalar product:

$$\langle\Phi|\Psi\rangle = \frac{1}{N^{0}}\frac{1}{V}\sum_{\mathbf{k}}\frac{\partial f_{\mathbf{k}}{}^{0}}{\partial E_{\mathbf{k}}{}^{0}}\Phi(\mathbf{k})\Psi(\mathbf{k}) \quad (2.4)$$

where N^0 is the normalization constant

$$N^0 = \frac{1}{V}\sum_{\mathbf{k}} \frac{\partial f_{\mathbf{k}}{}^0}{\partial E_{\mathbf{k}}{}^0} \tag{2.5}$$

We now expand the spin density in terms of eigenvectors $|\Psi_n(\mathbf{k}, \mathbf{q})\rangle$ of the operator $\hat{I}_{\mathbf{k}}^{(-)}(\mathbf{q})(\mathbf{1} - \hat{G}_{\mathbf{k}})$, where

$$\hat{I}_{\mathbf{k}}^{(-)}(\mathbf{q}) = i\hat{I}_{\mathbf{k}}^{(-)} + qu_z \tag{2.6}$$

and u_z is the z component of bogolon velocity

$$u_z = (\hbar k_z/m)\xi_{\mathbf{k}}/E_{\mathbf{k}}{}^0 \tag{2.7}$$

($\mathbf{q}$ is directed along the z axis). That is, we write

$$m(\mathbf{k}, \mathbf{q}, t) = \sum_{n=0}^{\infty} |\Psi_n(q, \mathbf{k})\rangle e^{-i\omega_n t} \tag{2.8}$$

and the kinetic equation takes the form

$$\omega_n|\Psi_n(\mathbf{k}, \mathbf{q})\rangle = \hat{I}_{\mathbf{k}}^{(-)}(\mathbf{q})(1 - \hat{G}_{\mathbf{k}})|\Psi_n(\mathbf{k}, \mathbf{q})\rangle \tag{2.9}$$

We have now reduced the linearized kinetic equation for the bogolon spin density to a linear eigenvalue equation.

We are interested in finding an expansion in powers of $\mathbf{q}$ of those frequencies that correspond to the hydrodynamic modes in the system. We remember that the hydrodynamic modes are those modes whose frequencies go to zero when $q \to 0$. Let us first introduce a new vector $|\chi_n(\mathbf{k}, \mathbf{q})\rangle$, defined by

$$|\chi_n(\mathbf{k}, \mathbf{q})\rangle = (1 - \hat{G}_{\mathbf{k}})|\Psi_n(\mathbf{k}, \mathbf{q})\rangle \tag{2.10}$$

and write Eq. (2.9) in the form

$$\omega_n(1 - \hat{G}_{\mathbf{k}})^{-1}|\chi_n(\mathbf{k}, \mathbf{q})\rangle = (i\hat{I}_{\mathbf{k}}^{(-)} + qu_z)|\chi_n(\mathbf{k}, \mathbf{q})\rangle \tag{2.11}$$

We now expand both ω_n and $|\chi_n(\mathbf{k}, \mathbf{q})\rangle$ in powers of q. Thus,

$$\omega_n = \omega_n^{(0)} + q\omega_n^{(1)} + q^2\omega_n^{(2)} + \cdots \tag{2.12}$$

and

$$|\chi_n(\mathbf{k}, \mathbf{q})\rangle = |\chi_n^{(0)}(\mathbf{k})\rangle + q|\chi_n^{(1)}(\mathbf{k})\rangle + q^2|\chi_n^{(2)}(\mathbf{k})\rangle + \cdots \tag{2.13}$$

We can systematically solve Eq. (2.11) for various terms in Eqs. (2.12) and (2.13).

In Eq. (I.4.31) we found that there is one hydrodynamic eigenfunction of $\hat{I}_{\mathbf{k}}^{(-)}$. We shall denote it $|\chi_1^{(0)}\rangle$ and define it by

$$|\chi_1^{(0)}\rangle = 1 \tag{2.14}$$

so it is normalized to one with respect to the scalar product in Eq. (2.4). The eigenfunction $|\chi_1^{(0)}\rangle$ has the property that

$$\hat{I}_{\mathbf{k}}^{(-)}|\chi_1^{(0)}\rangle = 0 \tag{2.15}$$

Furthermore, since $\hat{I}_{\mathbf{k}}^{(-)}$ is self-adjoint, the left eigenvector $\langle\chi_1^{(0)}|$ is equal to the right eigenvector $|\chi_1^{(0)}\rangle$. If we retain terms in Eq. (2.11) (for $n = 1$) to zeroth order in q and multiply by $\langle\chi_1^{(0)}|$, we find

$$\omega_1^{(0)} = 0 \tag{2.16}$$

If we retain terms to first order in q, we find that

$$\omega_1^{(1)} = \langle\chi_1^{(0)}|u_z|\chi_1^{(0)}\rangle / \langle\chi_1^{(0)}|(1 - \hat{G}_{\mathbf{k}})^{-1}|\chi_1^{(0)}\rangle = 0 \tag{2.17}$$

because of angle integrations in the matrix element $\langle\chi_1^{(0)}|u_z|\chi_1^{(0)}\rangle$. Furthermore, the first-order correction $|\chi_1^{(1)}(\mathbf{k})\rangle$ can be written

$$|\chi_1^{(1)}(\mathbf{k})\rangle = -(i\hat{I}_{\mathbf{k}}^{(-)})^{-1}u_z|\chi_1^{(0)}(\mathbf{k})\rangle \tag{2.18}$$

If we retain terms to second order in g, we obtain

$$\omega_1^{(2)} = -(\langle\chi_1^{(0)}|(1 - \hat{G}_{\mathbf{k}})^{-1}|\chi_1^{(0)}\rangle)^{-1}\langle\chi_1^{(0)}|u_z(i\hat{I}_{\mathbf{k}}^{(-)})^{-1}u_z|\chi_1^{(0)}\rangle \tag{2.19}$$

It is simple to show that

$$\langle\chi_1^{(0)}|(1 - \hat{G}_{\mathbf{k}})^{-1}|\chi_1^{(0)}\rangle = (1 - gN^0)^{-1} \tag{2.20}$$

where N^0 is defined in Eq. (2.5). Thus the hydrodynamic eigenvalue ω_1, to second order in q, can be written

$$\omega_1 = -q^2(1 - gN^0)\langle u_z|(i\hat{I}_{\mathbf{k}}^{(-)})^{-1}|u_z\rangle \tag{2.21}$$

If we compare Eq. (2.20) with Eq. (I.6.6) for the macroscopic dispersion relation of the diffusion mode, we obtain the following expression for the coefficient of spin diffusion D:

$$D = -(1 - gN^0)\langle u_z|(\hat{I}_{\mathbf{k}}^{(-)})^{-1}|u_z\rangle \tag{2.22}$$

Thus, the spin diffusion coefficient is given by a bogolon velocity autocorrelation function, as we expect.

3. SHEAR MODES

We now come to the more complicated problem of finding the frequencies of hydrodynamic modes that depend on the total bogolon density. In Eq. (I.4.22), we have written the kinetic equation for the total bogolon density $h(\mathbf{k}, \mathbf{q}, t)$. As for the case of the spin mode, we can expand $h(\mathbf{k}, \mathbf{q}, t)$ as follows:

$$h(\mathbf{k}, \mathbf{q}, t) = \sum_{n=1}^{\infty} |\Psi_n(\mathbf{k}, \mathbf{q})\rangle e^{-i\omega_n t} \tag{3.1}$$

The equation for $|\Psi_n(\mathbf{k}, \mathbf{q})\rangle$ takes the form

$$\omega_n|\Psi_n(\mathbf{k}, \mathbf{q})\rangle - qu_z[1 + \hat{J}_{\mathbf{k}}(\omega_n)]|\Psi_n(\mathbf{k}, \mathbf{q})\rangle = i\hat{I}_{\mathbf{k}}^{(+)}[1 + \hat{J}_{\mathbf{k}}(\omega_n)]|\Psi_n(\mathbf{k}, \mathbf{q})\rangle \tag{3.2}$$

where the collision operator $\hat{I}_{\mathbf{k}}^{(+)}$ is defined in terms of the collision operator $\hat{C}_{\mathbf{k}}^{(+)}$ [cf. Eq. (I.4.28)],

$$\hat{I}_{\mathbf{k}}^{(+)} = \frac{1}{\hbar}\left(\frac{\partial f_{\mathbf{k}}^0}{\partial E_{\mathbf{k}}^0}\right)^{-1} \hat{\mathbb{C}}_{\mathbf{k}}^{(+)} \tag{3.3}$$

and the integral operator $\hat{J}_{\mathbf{k}}(\omega_n)$ is defined by

$$\begin{aligned}
&\hat{J}_{\mathbf{k}}(\omega_n)|\Psi_n(\mathbf{k}, \mathbf{q})\rangle \\
&= \frac{q}{\omega_n}\left\{\frac{\hbar}{m}\frac{q}{\omega_n}|k_z\rangle - [1 + \Gamma(T)]^{-1}\left|\frac{\xi_{\mathbf{k}}}{E_{\mathbf{k}}^0}\right\rangle\right\}\left[\Gamma(T) + R(T)\frac{q^2}{\omega_n^2}\right]^{-1} \\
&\quad \times \frac{g\hbar}{m}\langle k_z'|\left\{\left(\frac{\xi_{\mathbf{k}'}}{E_{\mathbf{k}'}^0}\right)^2 - [1 + \Gamma(T)]\right\}|\Psi_n(\mathbf{k}', q)\rangle \\
&\quad + \left\{\frac{\hbar k_z}{m}\frac{q}{\omega_n} - [1 + \Gamma(T)]^{-1}\frac{\xi_{\mathbf{k}}}{E_{\mathbf{k}}^0}\right\} \\
&\quad \times \frac{1}{\omega_n}\frac{g}{2\Gamma(T)}\left\langle\frac{\xi_{\mathbf{k}'}}{E_{\mathbf{k}'}^0}\right| i\hat{I}_{\mathbf{k}'}^{(+)}\left|\Psi_n(\mathbf{k}', \mathbf{q})\right\rangle \\
&\quad + \left|\frac{\xi_{\mathbf{k}}}{E_{\mathbf{k}}^0}\right\rangle g[1 + \Gamma(T)]^{-1}\left\langle\frac{\xi_{\mathbf{k}'}}{E_{\mathbf{k}'}^0}\middle|\Psi_n(\mathbf{k}', \mathbf{q})\right\rangle \\
&\quad + \left|\frac{\xi_{\mathbf{k}}}{(E_{\mathbf{k}}^0)^2}\right\rangle\frac{g\Delta_0}{\chi(T)}\left\langle\frac{\Delta_0}{E_{\mathbf{k}'}^0}\middle|\Psi_n(\mathbf{k}', \mathbf{q})\right\rangle
\end{aligned} \tag{3.4}$$

The functions $\chi(T)$, $\Gamma(T)$, and $R(T)$ are defined as follows:

$$\chi(T) = 1 - \frac{g}{V}\sum_{\mathbf{k}}\left(\frac{\xi_{\mathbf{k}}}{E_{\mathbf{k}}^0}\right)^2\frac{1}{2E_{\mathbf{k}}^0}\tanh\frac{\beta E_{\mathbf{k}}^0}{2} \tag{3.5}$$

$$\Gamma(T) = -\frac{g}{V}\sum_{\mathbf{k}}\left(\frac{\Delta_0}{E_{\mathbf{k}}^0}\right)^2\frac{1}{E_{\mathbf{k}}^0}\tanh\frac{\beta E_{\mathbf{k}}^0}{2} \tag{3.6}$$

$$R(T) = -\frac{g}{V}\sum_{\mathbf{k}}\left(\frac{\hbar K_z}{m}\right)^2\frac{\partial f_{\mathbf{k}}^0}{\partial E_{\mathbf{k}}^0} - \frac{\rho g}{2m}[1 + \Gamma(T)] \tag{3.7}$$

where

$$f_{\mathbf{k}}^0 = [\exp(+\beta E_{\mathbf{k}}^0) + 1]^{-1} \tag{3.8}$$

In addition, $E_{\mathbf{k}}^0 = (\xi_{\mathbf{k}}^2 + \Delta_0^2)^{1/2}$ and $\xi_{\mathbf{k}} = (\hbar^2 k^2/2m) - \mu$ (μ is the chemical potential). Because $\hat{J}_{\mathbf{k}}(\omega_n)$ depends on the frequency ω_n, Eq. (3.2) is no longer a simple linear eigenvalue equation. However, we can still obtain a perturbation expansion for the frequencies ω_n.

As before, it is convenient to introduce a new function $|\chi_n(\mathbf{k}, \mathbf{q})\rangle$, which is defined by

$$|\chi_n(\mathbf{k}, \mathbf{q})\rangle = [1 + \hat{J}_{\mathbf{k}}(\omega_n)]|\Psi_n(\mathbf{k}, \mathbf{q})\rangle \tag{3.9}$$

Then, Eq. (3.2) can be written

$$\omega_n[1 + \hat{J}_{\mathbf{k}}(\omega_n)]^{-1}|\chi_n(\mathbf{k}, \mathbf{q})\rangle - qu_z|\chi_n(\mathbf{k}, \mathbf{q})\rangle = i\hat{I}_{\mathbf{k}}^{(+)}|\chi_n(\mathbf{k}, \mathbf{q})\rangle \tag{3.10}$$

The collision operator $\hat{I}_{\mathbf{k}}^{(+)}$ has four zero eigenfunctions, which we shall write

$$|\phi_1\rangle = B_1 k_x \tag{3.11a}$$

$$|\phi_2\rangle = B_2 k_y \tag{3.11b}$$

$$|\phi_3\rangle = B_3 k_z \tag{3.11c}$$

$$|\phi_4\rangle = B_4 E_{\mathbf{k}}^0 \tag{3.11d}$$

where B_α, $\alpha = 1,\dots, 4$, is a normalization constant,

$$B_\alpha = \langle \phi_\alpha | \phi_\alpha \rangle^{1/2} \tag{3.12}$$

Thus, the eigenfunctions $|\phi_\alpha\rangle$ are normalized to one. Note that since $\hat{I}_{\mathbf{k}}^{(+)}$ is self-adjoint, the left and right eigenvectors $|\phi_\alpha\rangle$ and $\langle\phi_\alpha|$ are equal.

Before we can perform a perturbation expansion, we must write $|\chi_{\alpha'}^{(0)}\rangle$ in terms of the correct linear combination of zero eigenfunctions $|\phi_\alpha\rangle$. That is, we write $|\chi_{\alpha'}^{(0)}\rangle$ in the form

$$|\chi_{\alpha'}^{(0)}\rangle = \sum_\alpha c_{\alpha'\alpha}|\phi_\alpha\rangle, \qquad \alpha, \alpha' = 1,\dots, 4 \tag{3.13}$$

and we must determine the coefficients $c_{\alpha'\alpha}$. We will anticipate some of the results in order to simplify the discussion. As for the case of the normal fluid, the shear modes will decouple from the longitudinal modes. This can be seen rather easily if we note that there are no terms in Eqs. (3.4) and (3.10) which can couple the eigenfunctions $|\phi_1\rangle$ and $|\phi_2\rangle$ to the eigenfunctions $|\phi_3\rangle$ and $|\phi_4\rangle$. Similarly, $|\phi_1\rangle$ and $|\phi_2\rangle$ cannot be coupled to one another. This enables us to make the following identification:

$$|\chi_1^{(0)}\rangle = |\phi_1\rangle \tag{3.14a}$$

$$|\chi_2^{(0)}\rangle = |\phi_2\rangle \tag{3.14b}$$

$$|\chi_3^{(0)}\rangle = c_{33}|\phi_3\rangle + c_{34}|\phi_4\rangle \tag{3.14c}$$

$$|\chi_4^{(0)}\rangle = c_{43}|\phi_3\rangle + c_{44}|\phi_4\rangle \tag{3.14d}$$

The eigenfunctions $|\chi_1^{(0)}\rangle$ and $|\chi_2^{(0)}\rangle$ correspond to the shear modes. The eigenfunctions $|\chi_3^{(0)}\rangle$ and $|\chi_4^{(0)}\rangle$ correspond to the longitudinal modes.

Let us now consider the shear modes and find expressions for ω_1 and ω_2 to second order in q. Since

$$\hat{J}_{\mathbf{k}}(\omega_\alpha)|\chi_\alpha^{(0)}\rangle \equiv 0 \qquad \text{for} \quad \alpha = 1, 2 \tag{3.15}$$

because of angle integrations, and because $\hat{I}_{\mathbf{k}}^{(+)}|\chi_\alpha^{(0)}\rangle = 0$, we find

$$\omega_\alpha^{(0)} = 0, \qquad \alpha = 1, 2 \tag{3.16}$$

The expression for $\omega_\alpha^{(1)}$, $\alpha = 1, 2$, can be written

$$\omega_\alpha^{(1)} = \langle\chi_\alpha^{(0)}|u_z|\chi_\alpha^{(0)}\rangle = 0, \qquad \alpha = 1, 2 \tag{3.17}$$

Note that $\omega_\alpha^{(1)}$ is zero because of angle integrations in the matrix element $\langle\chi_\alpha^{(0)}|u_z|\chi_\alpha^{(0)}\rangle$. The function $|\chi_\alpha^{(1)}(\mathbf{k})\rangle$, $\alpha = 1, 2$, can be written

$$|\chi_\alpha^{(1)}(\mathbf{k})\rangle = -(i\hat{I}_{\mathbf{k}}^{(+)})^{-1}u_z|\chi_\alpha^{(0)}(\mathbf{k})\rangle \tag{3.18}$$

If we retain terms of second order in q in Eq. (3.10), we finally obtain

$$\omega_\alpha^{(2)} = -\langle\chi_\alpha^{(0)}|u_z(i\hat{I}_{\mathbf{k}}^{(+)})^{-1}u_z|\chi_\alpha^{(0)}\rangle, \qquad \alpha = 1, 2 \tag{3.19}$$

and by symmetry $\omega_1^{(2)} = \omega_2^{(2)}$. Thus, to second order in q, the shear mode frequencies can be written

$$\omega_\alpha = -q^2\langle\chi_2^{(0)}|u_z(i\hat{I}_{\mathbf{k}}^{(+)})^{-1}u_z|\chi_\alpha^{(0)}\rangle, \qquad \alpha = 1, 2 \tag{3.20}$$

If we now compare Eq. (3.20) with Eq. (I.6.21) for the macroscopic dispersion relation for the same shear modes, we obtain the following microscopic expression for the coefficient of shear viscosity:

$$\eta = -\langle k_x|k_x\rangle^{-1}\rho_n{}^0\langle k_x u_z|(\hat{I}_{\mathbf{k}}^{(+)})^{-1}|k_x u_z\rangle \tag{3.21}$$

As we expect, the coefficient of shear viscosity is expressed in terms of a bogolon momentum current correlation function.

4. LONGITUDINAL MODES—GENERAL DISCUSSION

We now wish to obtain dispersion relations for the longitudinal modes. We must find four of them, two first-sound modes and two second-sound modes. If Eq. (3.6) were a simple linear eigenvalue equation, this would not be possible. However, the dependence of Eq. (3.6) on $\hat{J}_{\mathbf{k}}(\omega_n)$ changes its whole character. For the shear modes $\hat{J}_{\mathbf{k}}(\omega_n)$ does not contribute, but for the longitudinal modes it plays a very important role.

In I we were able to close the kinetic equation for the total bogolon density by requiring that the total particle density satisfy a continuity equation (the usual assumption of two-fluid hydrodynamics). This enabled us to express the chemical potential in terms of bogolon distribution functions, thereby closing the kinetic equations. However, it also introduced an unphysical imaginary term into the chemical potential. It is easy to see that unless this imaginary term can be neglected, there are no microscopic longitudinal hydrodynamic modes. The imaginary term in the chemical potential is the term in Eq. (3.4) that depends on the collision integral. Let us consider the effect of this term on the operator $[1 + \hat{J}_{\mathbf{k}}(\omega_\alpha)]^{-1}$. We can write $\hat{J}_{\mathbf{k}}(\omega_\alpha)$ as follows:

$$\hat{J}_{\mathbf{k}}(\omega_\alpha) = \hat{M}_{\mathbf{k}} + \hat{N}_{\mathbf{k}} + \frac{q^2 \hat{A}_{\mathbf{k}}}{\Gamma(T)\omega_\alpha^2 + R(T)q^2} + \frac{q\omega_\alpha \hat{B}_{\mathbf{k}}}{\Gamma(T)\omega_\alpha^2 + R(T)q^2} + \frac{q\hat{K}_1}{\omega_\alpha^2} + \frac{\hat{K}_2}{\omega_\alpha} \tag{4.1}$$

where

$$\hat{A}_{\mathbf{k}} = g\left(\frac{\hbar}{m}\right)^2 |k_z\rangle\langle k_z| \left\{\left(\frac{\xi_{\mathbf{k}}}{E_{\mathbf{k}}^0}\right)^2 - [1 + \Gamma(T)]\right\} \tag{4.2}$$

$$\hat{B}_{\mathbf{k}} = -g\left(\frac{\hbar}{m}\right)[1 + \Gamma(T)]^{-1} \left|\frac{\xi_{\mathbf{k}}}{E_{\mathbf{k}}^0}\right\rangle\langle k_z| \left\{\left(\frac{\xi_{\mathbf{k}}}{E_{\mathbf{k}}^0}\right)^2 - [1 + \Gamma(T)]\right\} \tag{4.3}$$

$$\hat{K}_1 = \frac{\hbar}{m}\frac{g}{2\Gamma(T)} |k_z\rangle\left\langle\frac{\xi_{\mathbf{k}}}{E_{\mathbf{k}}^0}\right| i\hat{I}_k^{(+)} \tag{4.4}$$

$$\hat{K}_2 = \frac{g}{2\Gamma(T)} [1 + \Gamma(T)]^{-1} \left|\frac{\xi_{\mathbf{k}}}{E_{\mathbf{k}}^0}\right\rangle\left\langle\frac{\xi_{\mathbf{k}}}{E_{\mathbf{k}}^0}\right| i\hat{I}_{\mathbf{k}}^{(+)} \tag{4.5}$$

$$\hat{M}_{\mathbf{k}} = g[1 + \Gamma(T)]^{-1} \left|\frac{\xi_{\mathbf{k}}}{E_{\mathbf{k}}^0}\right\rangle\left\langle\frac{\xi_{\mathbf{k}}}{E_{\mathbf{k}}^0}\right| \tag{4.6}$$

$$\hat{N}_{\mathbf{k}} = \frac{g\Delta_0^2}{\chi(T)} \left|\frac{\xi_{\mathbf{k}}}{(E_{\mathbf{k}}^0)^2}\right\rangle\left\langle\frac{\Delta_0}{E_{\mathbf{k}}^0}\right| \tag{4.7}$$

Let us attempt to find $\omega_\alpha^{(0)}$ by retaining terms in Eq. (3.6) which are zeroth order in q. We then obtain the following equation for $\omega_\alpha^{(0)}$:

$$\omega_\alpha^{(0)}\left\langle \phi_{\alpha'}\right|\left(1 + \hat{M}_{\mathbf{k}} + \hat{N}_{\mathbf{k}} + \frac{1}{\omega_\alpha^{(0)}}\hat{K}_2\right)^{-1}\left|\chi_\alpha^{(0)}\right\rangle = 0 \tag{4.8}$$

It is easy to evaluate various matrix elements in Eq. (4.8). Let us first note that because of angle integrations,

$$(\hat{M}_{\mathbf{k}} + \hat{N}_{\mathbf{k}})|\phi_3\rangle = \langle\phi_3|(\hat{M}_{\mathbf{k}} + \hat{N}_{\mathbf{k}} + \hat{K}_2) = 0 \tag{4.9}$$

Furthermore, if we neglect contributions to Eq. (4.8) that are of order Δ_0/E_F, where E_F is the energy of the Fermi surface, we find

$$\langle\phi_4|(\hat{M}_{\mathbf{k}} + \hat{N}_{\mathbf{k}} + \hat{K}_2) \approx \hat{M}_{\mathbf{k}}|\phi_4\rangle \approx 0 \tag{4.10}$$

(these terms involve integrals odd in $\xi_{\mathbf{k}}$), and finally, because of conservation of bogolon momentum and energy, we have

$$\hat{K}_2|\phi_4\rangle = \hat{K}_2|\phi_3\rangle = 0 \tag{4.11}$$

If we make use of these results in Eq. (4.8), we find

$$\omega_\alpha^{(0)} = 0 \tag{4.12}$$

However, we now see that our perturbation expansion is inconsistent because higher order terms depend on $(1/\omega_\alpha^{(0)})\hat{K}_1$. If $\omega_\alpha^{(0)} = 0$, then higher order terms are infinite. Thus the longitudinal hydrodynamic modes appear to be undefined. Note, however, that if we relax condition (4.10), we no longer obtain the result $\omega_\alpha^{(0)} = 0$. Then $\omega_\alpha^{(0)}$ can have a finite value, and our perturbation expansion can be well defined. Equation (4.10) is one of the assumptions we made in deriving the closed kinetic equations. We cannot relax it here without changing the kinetic equations. For small q, the imaginary terms contribute through matrix elements of the form $\langle\xi_{\mathbf{k}}|\hat{K}_1|\xi_{\mathbf{k}}\rangle$ and $\langle\xi_{\mathbf{k}}|\hat{K}_2|\xi_{\mathbf{k}}\rangle$. Since $\xi_{\mathbf{k}}$ is zero at the Fermi surface and $\partial f_{\mathbf{k}}{}^0/\partial E_{\mathbf{k}}{}^0$ is fairly sharply peaked at the Fermi surface, the contributions from these matrix elements are small.

The imaginary terms appear to originate in the approximations that were made in deriving the scalar bogolon equations from the kinetic matrix describing particle propagation, and they act as a small-particle source. The nonconservation of particle number by the bogolon equations is a drawback of this model, and one which has been mentioned before.[2,3] In the next section, we shall neglect these small terms, since this appears to be consistent with the approximations we have already made in deriving the scalar bogolon equations.

5. LONGITUDINAL MODES—DISPERSION RELATIONS

Let us now find the dispersion relations for the longitudinal modes to second order in q. The kinetic equation can be written in the form

$$\omega_\alpha[1 + \hat{P}_{\mathbf{k}}(\omega_\alpha)]^{-1}|\chi_\alpha(\mathbf{k}, \mathbf{q})\rangle - qu_z|\chi_\alpha(\mathbf{k}, \mathbf{q})\rangle = i\hat{I}_{\mathbf{k}}^{(+)}|\chi_\alpha(\mathbf{k}, \mathbf{q})\rangle \tag{5.1}$$

where

$$\hat{P}_{\mathbf{k}}(\omega_\alpha) = \hat{M}_{\mathbf{k}} + \hat{N}_{\mathbf{k}} + \hat{A}_{\mathbf{k}}(\omega_\alpha) + \hat{B}_{\mathbf{k}}(\omega_\alpha) \tag{5.2}$$

$$\hat{A}_{\mathbf{k}}(\omega_\alpha) = q^2\hat{A}_{\mathbf{k}}/[\omega_\alpha{}^2\Gamma(T) + R(T)q^2] \tag{5.3}$$

and

$$\hat{B}_{\mathbf{k}}(\omega_\alpha) = q\omega_\alpha\hat{B}_{\mathbf{k}}/[\omega_\alpha{}^2\Gamma(T) + R(T)q^2] \tag{5.4}$$

(cf. Section 4 for a discussion). In the Appendix we show that

$$\langle\phi_4|[1+\hat{P}_{\mathbf{k}}(\omega_\alpha)]^{-1}|\phi_4\rangle = 1 \tag{5.5}$$

$$\langle\phi_4|[1+\hat{P}_{\mathbf{k}}(\omega_\alpha)]^{-1}|\phi_3\rangle = 0 \tag{5.6}$$

$$\langle\phi_3|[1+\hat{P}_{\mathbf{k}}(\omega_\alpha)]^{-1}|\phi_4\rangle = 0 \tag{5.7}$$

$$\langle\phi_3|[1+\hat{P}_{\mathbf{k}}(\omega_\alpha)]^{-1}|\phi_3\rangle = \langle\phi_3|[1+\hat{A}_{\mathbf{k}}(\omega_\alpha)]^{-1}|\phi_3\rangle \tag{5.8}$$

$$\langle\phi_4|[1+\hat{P}_{\mathbf{k}}(\omega_\alpha)]^{-1}|\chi_\alpha^{(1)}\rangle = \langle\phi_4|\chi_\alpha^{(1)}\rangle \tag{5.9}$$

and

$$\langle\phi_3|[1+\hat{P}_{\mathbf{k}}(\omega_\alpha)]^{-1}|\chi_\alpha^{(1)}\rangle = \langle\phi_3|[1+\hat{A}_{\mathbf{k}}(\omega_\alpha)]^{-1}|\chi_\alpha^{(1)}\rangle \tag{5.10}$$

From the discussion in Section 4 (neglecting $\hat{K}_1$ and $\hat{K}_2$), we find immediately that

$$\omega_\alpha^{(0)} = 0 \tag{5.11}$$

It is now useful to expand $[1+\hat{A}_{\mathbf{k}}(\omega_\alpha)]^{-1}$ in powers of q. If we use Eq. (5.11), we find

$$[1+\hat{A}_{\mathbf{k}}(\omega_\alpha)]^{-1} = \left[1+\frac{\hat{A}_{\mathbf{k}}}{(\omega_\alpha^{(1)})^2\Gamma+R}\right]^{-1}\left\{1-q\frac{2\omega_\alpha^{(1)}\omega_\alpha^{(2)}\hat{A}_{\mathbf{k}}}{[(\omega_\alpha^{(1)})^2\Gamma+R]^2}+\cdots\right\} \tag{5.12}$$

We can now use Eqs. (5.5)–(5.8) and (5.12) to obtain the frequencies $\omega_\alpha^{(1)}$.

Let us retain terms in Eq. (3.6) that are of order q,

$$\omega_\alpha^{(1)}[1+\hat{A}_{\mathbf{k}}(\omega_\alpha)]_0^{-1}|\chi_\alpha^{(0)}\rangle - u_z|\chi_\alpha^{(0)}\rangle = i\hat{I}_{\mathbf{k}}^{(+)}|\chi_\alpha^{(1)}\rangle \tag{5.13}$$

where

$$[1+\hat{A}_{\mathbf{k}}(\omega_\alpha)]_0^{-1} = \left[1+\frac{\hat{A}_{\mathbf{k}}}{(\omega_\alpha^{(1)})^2\Gamma+R}\right]^{-1} \tag{5.14}$$

We see immediately that

$$|\chi_\alpha^{(1)}\rangle = (i\hat{I}_{\mathbf{k}}^{(+)})^{-1}\{\omega_\alpha^{(1)}[1+\hat{A}_{\mathbf{k}}(\omega_\alpha)]_0^{-1} - u_z\}|\chi_\alpha^{(0)}\rangle \tag{5.15}$$

Let us now multiply Eq. (5.13) by $\langle\phi_{\alpha''}|$ and use Eqs. (3.10c)–(3.10d). We find

$$\omega_\alpha^{(1)}\sum_{\alpha=3,4} c_{\alpha\alpha'}\langle\phi_{\alpha''}|[1+\hat{A}_k(\omega_\alpha)]_0^{-1}|\phi_{\alpha'}\rangle - \sum_{\alpha=3,4} c_{\alpha\alpha'}\langle\phi_{\alpha''}|u_z|\phi_{\alpha'}\rangle = 0 \tag{5.16}$$

If we let $\alpha'' = 3$ and use Eqs. (5.7) and (5.8), we find

$$\omega_\alpha^{(1)}c_{\alpha 3}\langle\phi_3|[1+\hat{A}_{\mathbf{k}}(\omega_\alpha)]_0^{-1}|\phi_3\rangle - c_{\alpha 4}\langle\phi_3|u_z|\phi_4\rangle = 0 \tag{5.17}$$

while for $\alpha'' = 4$ we find

$$\omega_\alpha^{(1)} c_{\alpha 4} - c_{\alpha 3}\langle \phi_4 | u_z | \phi_3 \rangle = 0 \tag{5.18}$$

The matrix element $\langle \phi_3 | [1 + \hat{A}_{\mathbf{k}}(\omega_\alpha)]_0^{-1} | \phi_3 \rangle$ is easy to evaluate. We obtain

$$\langle \phi_3 | [1 + \hat{A}_{\mathbf{k}}(\omega_\alpha)]_0^{-1} | \phi_3 \rangle = \frac{(\omega_\alpha^{(1)})^2 \Gamma + R}{(\omega_\alpha^{(1)})^2 \Gamma + R + (g\hbar^2/m^2)\langle k_z | \{(\xi_{\mathbf{k}}/E_{\mathbf{k}}{}^0)^2 - [1 + \Gamma(T)]\} | k_z \rangle} \tag{5.19}$$

From Eqs. (5.17) and (5.18), we obtain the following equation for $\omega_\alpha^{(1)}$:

$$\begin{aligned} &\Gamma(T)[\omega_\alpha^{(1)}]^4 + [\omega_\alpha^{(1)}]^2 [R(T) - \Gamma(T)\langle \phi_3 | u_z | \phi_4 \rangle \langle \phi_4 | u_z | \phi_3 \rangle] \\ &\quad - \langle \phi_3 | u_z | \phi_4 \rangle \langle \phi_4 | u_z | \phi_3 \rangle \\ &\quad \times \left[R(T) + g\left(\frac{\hbar}{m}\right)^2 \left\langle k_z \middle| \left\{ \left(\frac{\xi_{\mathbf{k}}}{E_{\mathbf{k}}{}^0}\right)^2 - [1 + \Gamma(T)] \right\} \middle| k_z \right\rangle \right] = 0 \end{aligned} \tag{5.20}$$

Equation (5.20) has four solutions, and therefore we obtain the desired four longitudinal sound modes, two first-sound modes and two second-sound modes. It is interesting to compare Eq. (5.20) to Eq. (I.6.30). We then obtain the following microscopic expressions for the various sound velocities:

$$c_s{}^2 = \langle \phi_3 | u_z | \phi_4 \rangle \langle \phi_4 | u_z | \phi_3 \rangle \tag{5.21}$$

$$u_T{}^2 = -R(T)/\Gamma(T) \tag{5.22}$$

$$\frac{(c_T{}^2 - c_s{}^2)u_T{}^2}{(c_s{}^2 - u_T{}^2)^2} = \frac{-\langle \phi_3 | u_z | \phi_4 \rangle \langle \phi_4 | u_z | \phi_3 \rangle g(\hbar/m)^2 \langle k_z | \{(\xi_{\mathbf{k}}/E_{\mathbf{k}}{}^0)^2 - [1 + \Gamma(T)]\} | k_z \rangle}{[\langle \phi_3 | u_z | \phi_4 \rangle \langle \phi_4 | u_z | \phi_3 \rangle \Gamma(T) + R(T)]^2} \tag{5.23}$$

It is interesting to note that although there is no continuity equation for the bogolons, there is still an adiabatic sound velocity, but its value is determined in terms of the energy and momentum eigenfunctions. In the normal fluid,[3] there is a similar contribution to the adiabatic sound velocity, but at low temperature it is far less important than the contribution that results from particle conservation in the normal fluid. If bogolons were conserved during collisions, there would be additional important contributions to all these velocities. In terms of the above velocities, the frequency of the first-sound mode may be written

$$\omega_s = \frac{q}{\sqrt{2}} \left[c_s{}^2 + u_T{}^2 + (c_s{}^2 - u_T{}^2)\left(1 - \frac{4(c_T{}^2 - c_s{}^2)u_T{}^2}{(c_s{}^2 - u_T{}^2)^2}\right)^{1/2} \right]^{1/2} \tag{5.24}$$

and

$$\omega_T = \frac{q}{\sqrt{2}}\left[c_s^2 + u_T^2 - (c_s^2 - u_T^2)\left(1 - \frac{4(c_T^2 - c_s^2)u_T^2}{c_s^2 - u_T^2}\right)^{1/2}\right]^{1/2} \quad (5.25)$$

We see that because of the presence of $P_{\mathbf{k}}(\omega_\alpha)$ in the equation for the total bogolon density, the dimension of the space of longitudinal hydrodynamic modes has increased from the expected two to four. The fact that we have obtained four longitudinal modes is not an accident. It is a result of the peculiar structure of $P_{\mathbf{k}}(\omega_\alpha)$.

We now have four longitudinal hydrodynamic modes. We shall denote their zero eigenfunctions by $|\chi_\beta^{(0)}\rangle$, where $\beta = a, b, c, d$. We know that $\omega_\beta^{(0)} = 0$. We will let $\beta = a$ and b denote the first-sound modes and $\beta = c$ and d denote the second-sound modes. Thus,

$$\omega_a^{(1)} = +\omega_s, \qquad \omega_b^{(1)} = -\omega_s, \qquad \omega_c^{(1)} = +\omega_T, \qquad \omega_d^{(1)} = -\omega_T \quad (5.26)$$

The eigenfunctions $|\chi_\beta^{(0)}\rangle$ are not linearly independent. They are expressed in terms of $|\phi_3\rangle$ and $|\phi_4\rangle$ according to the equation

$$|\chi_\beta^{(0)}\rangle = \sum_{\alpha=3,4} c_{\beta\alpha}|\phi_\alpha\rangle \quad (5.27)$$

Equation (5.1) can now be written in the form

$$\omega_\beta^{(1)} \sum_{\alpha'=3,4} c_{\beta\alpha'}\langle\phi_{\alpha''}|[1 + \hat{A}_k(\omega_\beta)]_0^{-1}|\phi_{\alpha'}\rangle = \sum_{\alpha'=3,4} c_{\beta\alpha'}\langle\phi_{\alpha''}|u_z|\phi_{\alpha'}\rangle \quad (5.28)$$

and we can solve for the coefficients $c_{\beta\alpha'}$. We find that

$$c_{\beta 3} = c_{\beta 4}\frac{\langle\phi_3|u_z|\phi_4\rangle}{\omega_\beta^{(1)}\langle\phi_3|[1 + \hat{A}_{\mathbf{k}}(\omega_\beta)]_0^{-1}|\phi_3\rangle} \quad (5.29)$$

We can completely determine the coefficients $c_{\beta\alpha}$ by requiring that the functions $|\chi_\beta^{(0)}\rangle$ be normalized to one,

$$\langle\chi_\beta^{(0)}|\chi_\beta^{(0)}\rangle = 1 \quad (5.30)$$

We will not write all the coefficients here.

Let us next obtain an expression for $\omega_\beta^{(2)}$. We can do this if we retain terms in Eq. (5.1) of order q^2. We then find

$$\omega_\beta^{(1)}[1 + \hat{A}_{\mathbf{k}}(\omega_\beta)]_0^{-1}|\chi_\beta^{(1)}\rangle + \omega_\beta^{(2)}[1 + \hat{A}_{\mathbf{k}}(\omega_\beta)]_{(1)}^{-1}|\chi_\beta^{(0)}\rangle + \omega_\beta^{(2)}[1 + \hat{A}_{\mathbf{k}}(\omega_\beta)]_0^{-1}|\chi_\beta^{(0)}\rangle - u_z|\chi_\beta^{(1)}\rangle = i\hat{\Gamma}_{\mathbf{k}}^{(+)}|\chi_\beta^{(2)}\rangle \quad (5.31)$$

where

$$[1 + \hat{A}(\omega_\beta)]_{(1)}^{-1} = -[1 + \hat{A}_{\mathbf{k}}(\omega_\beta)]_{(0)}^{-1}\frac{2(\omega_\beta^{(1)})^2\hat{A}_{\mathbf{k}}}{[(\omega_\beta^{(1)})^2\Gamma + R]^2} \quad (5.32)$$

If we multiply by $\langle\chi_\beta^{(0)}|$, we obtain the following expression for $\omega_\beta^{(2)}$:

$$\omega_\beta^{(2)} = \{\langle\chi_\beta^{(0)}|[1 + \hat{A}_{\mathbf{k}}(\omega_\beta)]_{(1)}^{-1}|\chi_\beta^{(0)}\rangle + \langle\chi_\beta^{(0)}|[1 + \hat{A}_{\mathbf{k}}(\omega_\beta)]_{(0)}^{-1}|\chi_\beta^{(0)}\rangle\}^{-1} \times (\langle\chi_\beta^{(0)}|\{u_z - \omega_\beta^{(1)}[1 + \hat{A}_{\mathbf{k}}(\omega_\beta)]_{(0)}^{-1}\}|\chi_\beta^{(1)}\rangle) \tag{5.33}$$

where $|\chi_\beta^{(1)}\rangle$ has been given in Eq. (5.15). Thus, the microscopic expression for the longitudinal modes takes the form

$$\omega_\beta = q\omega_\beta^{(1)} + q^2\omega_\beta^{(2)} + \cdots \qquad (\beta = a, b, c, d) \tag{5.34}$$

as we expect. We have thus obtained microscopic expressions for all hydrodynamic modes directly from the kinetic equation, without having to solve the quartic equation which results from Eqs. (I.6.26)–(I.6.29). However, in order to obtain microscopic expressions for the longitudinal transport coefficients from this method, we must solve the quartic equation to order q^2. We can then match the solutions obtained to the microscopic equations for the hydrodynamic frequencies and thereby obtain microscopic expressions for the longitudinal transport coefficients.

6. CONCLUDING REMARKS

The microscopic expressions we have obtained for the amplitude and dispersion relations of the hydrodynamic modes will be good as long as there are no other eigenvalues of the collision operators which are close to zero. The condition that the perturbation expansion be valid is that

$$|\langle\chi_n^{(0)}|\hat{I}_{\mathbf{k}}^{(+)}|\chi_n^{(0)}\rangle| \gg q\langle\chi_n|u_z|\chi_\alpha\rangle \tag{6.1}$$

where $|\chi_n^{(0)}\rangle$ is a nonhydrodynamic eigenfunction of $\hat{I}_{\mathbf{k}}^{(+)}$. From Ref. 2, we know that for the normal phase the bogolon excitations change to particle-like excitations and an additional hydrodynamic eigenfunction of $\hat{I}_{\mathbf{k}}^{(+)}$ appears due to particle conservation. Thus, even in the superfluid phase we expect that there will be one eigenfunction of $\hat{I}_{\mathbf{k}}^{(+)}$ with an eigenvalue much smaller than the rest. This will correspond to a mode whose amplitude is slowly damped due to collisions. This mode will have no effect on the spin diffusion mode and probably little or no effect on the shear modes, since it will be largely orthogonal to them, but it can have an important effect on the longitudinal modes. However, since we can choose q as small as we want, we can in principle always satisfy Eq. (6.1) for inhomogeneities of sufficiently long wavelengths.

APPENDIX

In this appendix, we shall derive Eqs. (5.5)–(5.10). Let us first note the following facts. Because of angle integrations the following quantities are identically zero:

$$(\hat{B} + \hat{A})(\hat{B} + \hat{M} + \hat{N}) = 0 \tag{A.1}$$

$$(\hat{M} + \hat{N})\hat{A} = 0 \tag{A.2}$$

$$(\hat{M} + \hat{N})|\phi_3\rangle = 0 \tag{A.3}$$

$$\langle\phi_3|(\hat{M} + \hat{N} + \hat{B}) = 0 \tag{A.4}$$

$$(\hat{A} + \hat{B})|\phi_4\rangle = 0 \tag{A.5}$$

$$\langle\phi_4|\hat{A} = 0 \tag{A.6}$$

In deriving the bogolon kinetic equations, we neglected terms of order Δ_0/E_F, where E_F is the energy at the Fermi surface. We will also neglect these terms here. Then we find

$$\langle\phi_4|(\hat{M} + \hat{N} + \hat{B}) \approx 0 \tag{A.7}$$

$$\hat{M}|\phi_4\rangle \approx 0 \tag{A.8}$$

$$\hat{N}(\hat{M} + \hat{B}) \approx 0 \tag{A.9}$$

We can now evaluate matrix elements of $[1 + \hat{P}_{\mathbf{k}}(\omega_\alpha)]^{-1}$.

From Eqs. (A.6) and (A.9), we find immediately

$$\langle\phi_4|(1 + \hat{M} + \hat{N} + \hat{A} + \hat{B})^{-1}|\phi_3\rangle = 0 \tag{A.10}$$

$$\langle\phi_4|(1 + \hat{M} + \hat{N} + \hat{A} + \hat{B})^{-1}|\phi_4\rangle = 1 \tag{A.11}$$

$$\langle\phi_4|(1 + \hat{M} + \hat{N} + \hat{A} + \hat{B})^{-1}|\chi_\alpha^{(1)}\rangle = \langle\phi_4|\chi_\alpha^{(1)}\rangle \tag{A.12}$$

Let us next consider $\langle\phi_3|(1 + \hat{P}_{\mathbf{k}})^{-1}|\phi_4\rangle$. From Eqs. (A.1), (A.4), and (A.5), we find

$$\begin{aligned}
&\langle\phi_3|(1 + \hat{A} + \hat{B} + \hat{M} + \hat{N})^{-1}|\phi_4\rangle \\
&\quad = \langle\phi_3|[1 + (1 + \hat{A} + \hat{B})^{-1}(\hat{M} + \hat{N})]^{-1}(1 + \hat{A} + \hat{B})^{-1}|\phi_4\rangle \\
&\quad = \langle\phi_3|(1 + \hat{M} + \hat{N})^{-1}|\phi_4\rangle = 0
\end{aligned} \tag{A.13}$$

If we use Eqs. (A.1) and (A.4), the matrix element $\langle\phi_3|(1 + \hat{P}_{\mathbf{k}})^{-1}|\phi_2\rangle$ can be written

$$\begin{aligned}
&\langle\phi_3|(1 + \hat{A} + \hat{B} + \hat{M} + \hat{N})^{-1}|\phi_3\rangle \\
&\quad = \langle\phi_3|[1 + (1 + \hat{A})^{-1}(M + N + B)]^{-1}(1 + A)^{-1}|\phi_3\rangle \\
&\quad = \langle\phi_3|(1 + \hat{M} + \hat{N} + \hat{B})^{-1}(1 + \hat{A})^{-1}|\phi_3\rangle \\
&\quad = \langle\phi_3|(1 + \hat{A})^{-1}|\phi_3\rangle
\end{aligned} \tag{A.14}$$

By a simpler calculation we also obtain

$$\langle\phi_3|(1+\hat{A}+\hat{B}+\hat{M}+\hat{N})^{-1}|\chi_\alpha^{(1)}\rangle = \langle\phi_3|(1+\hat{A})^{-1}|\chi_\alpha^{(1)}\rangle \quad \text{(A.15)}$$

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